Nonlinear Free Flexural Vibration of Thin Circular Cylindrical Shells

T. K. Varadan*
Indian Institute of Technology, Madras, India
G. Prathap†
National Aeronautical Laboratory, Bangalore, India
and
H. V. Ramani†
Defence Research and Development Laboratory, Hyderabad, India

Introduction

THE early development on the topic of nonlinear vibrations of isotropic circular cylindrical shells is well documented by Evensen.[1] In 1976, Raju and Rao[2] presented a finite-element solution, and Evensen[3] commented that the authors[2] had ignored the physics of the problem that thin shells bend more readily than they stretch. Subsequently, Prathap[4] pointed out some inconsistencies in the mathematical analysis carried out by Evensen[5] and also in the physical behavior of the three-term model of Dowell and Ventres.[6]

The comments made in Ref. 4 led to reinvestigation of the earlier problem in the present study. The axisymmetric part of the assumed deflected shape plays an important role in the nonlinear behavior of the shell, and so two appropriate three-term mode shapes for the transverse displacement are chosen. The modal equations obtained by the Galerkin method are solved by the fourth-order Runge-Kutta method to obtain the amplitude-frequency relationship. The numerical results based on the present study and on the analysis of Evensen[5] are compared with the existing experimental values.

Modal Equations

The nonlinearity arising from the stretching of the median surface is considered, and the approximations of Donnell's shallow shell theory are used. The governing equations for the free flexural vibrations of a thin isotropic circular cylindrical shell undergoing moderately large amplitude of vibration are

$$(1/Eh)\nabla^4 F = -(w_{,xx}/a) + w^2{}_{,xy} - w_{,xx}w_{,yy} \tag{1}$$

$$D\nabla^4 w = F_{,yy}w_{,xx} + F_{,xx}w_{,yy} - 2F_{,xy}w_{,xy} + (F_{,xx}/a) - \rho h w_{,tt} \tag{2}$$

where a is the radius, h the thickness, F the stress function, w the normal displacement that is positive in the inward direction, and ρ the mass density.

The governing equations of motion, Eqs. (1) and (2), are solved approximately. Even though the Galerkin method is a powerful approximate method, its results are highly dependent on the assumed vibration mode shape. The axisymmetric part in the assumed vibration mode shape plays an important role in the prediction of the vibrational behavior. The present investigation lays emphasis on the choice of mode shape. A three-term mode shape, as

$$w(x,y,t) = A(t)\sin(m\pi x/L)\cos(ny/a) + B(t)\sin(m\pi x/L)\sin(ny/a) + C(t)\sin^2(m\pi x/L) \tag{3}$$

where A, B, and C are three independent degrees of freedom, is chosen. Here, L is the length of the shell, and m and n are the number of axial (x) half-waves and circumferential (y) waves, respectively. The axisymmetric part of the mode shape satisfies certain physical requirements that thin shells bend more readily than they stretch. At large amplitudes, the extensional and flexural modes couple in such a way as to reduce the extensional strain energy due to stretching involved at large amplitude of vibration. Also, the inextensionality assumption need not be enforced here, unlike Ref. 5, because an independent generalized coordinate is chosen for the axisymmetric part of the mode shape.

In addition to the one already mentioned, a second vibration mode shape,

$$w(x,y,t) = A(t)\sin(\pi x/L)\cos(ny/a) + B(t)\sin(\pi x/L)\sin(ny/a) + C(t)\sin(\pi x/L) \tag{4}$$

is also examined. This mode shape is exactly the same as the one suggested in Ref. 6 when $m=1$, and the axisymmetric part is a predominantly contractional mode. This mode shape is considered to see whether the results based on this mode shape agree with the experimental results.

The shell is considered simply supported at both ends and immovable in the in-plane directions. It can be seen that the chosen mode shapes satisfy the boundary condition $w=0$ at edges $x=0$, L. However, the moment boundary condition is not satisfied by the mode shape given by Eq. (3), whereas it is satisfied by Eq. (4).

The assumed vibration mode shapes for w are substituted into the right-hand side of the compatibility equation, Eq. (1). The relevant expression for the stress function F is evaluated, and the continuity and in-plane conditions are satisfied on an average. Substitution of the expressions for F and w in the differential equation of motion, Eq. (2), and then application of the Galerkin's technique yields the three modal equations in the nondimensional form as

$$(d^2a_{mo}/d\tau^2) + d_1 a_{mo}^3 \times d_2 a_{mo}(a_{mn}^2 + b_{mn}^2) + d_4 a_{mo}^2 + d_5(a_{mn}^2 + b_{mn}^2) + d_7 a_{mo} = 0$$

$$(d^2a_{mn}/d\tau^2) + d_8 a_{mn}(a_{mn}^2 + b_{mn}^2) + d_9 a_{mn}a_{mo}^2 + d_{11}a_{mn}a_{mo} + d_{12}a_{mn} = 0$$

$$(d^2b_{mn}/d\tau^2) + d_8 b_{mn}(a_{mn}^2 + b_{mn}^2) + d_9 b_{mn}a_{mo}^2 + d_{11}b_{mn}a_{mo} + d_{12}b_{mn} = 0$$

where τ is the nondimensional time $(D\pi^4/\rho hL^4)^{1/2}t = \omega_{mn}t$ and a_{mo}, a_{mn}, and b_{mn} are the nondimensional amplitudes equal to C/h, A/h, and B/h, respectively. The coefficients d_1–d_{12} for both assumed values of w are given in Ref. 7. One can see that the modal equations have only nonlinear elasticity terms, and they arise from the stretching of the median surface due to large rotation terms, in accordance with the von Kármán plate theory. This is similar to the familiar flat plate and shallow spherical shell problems, which have only nonlinear elasticity terms. The modal equation given in Ref. 5 has both nonlinear inertia and nonlinear elasticity terms. The significant nonlinear term therein is nonlinear inertia terms in contrast with the present analysis.

Numerical Results and Conclusions

The modal equations are integrated numerically using the fourth-order Runge-Kutta method to obtain the amplitude-period relationship. The initial conditions for the free vibration study carried out here are $a_{mo} = a_{mn} = b_{mn} = 0$, $da_{mo}/d\tau = db_{mn}/d\tau = 0$, $da_{mn}/d\tau \neq 0$ at $\tau = 0$. The change to different initial conditions (amplitude at $\tau = 0$, $A_{mn} \neq 0$ practically did not alter the final results. Also, the time-step size taken is such that further reduction in time-step size does not alter the results.

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*Professor, Department of Aerospace Engineering.

†Scientist, Structures Division.

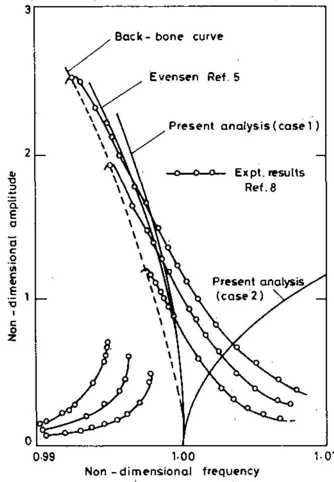

Fig. 1 Comparison with Olson's shell ($\xi = 0.1635$, $\mathcal{E} = 0.003025$, $\nu = 0.365$).

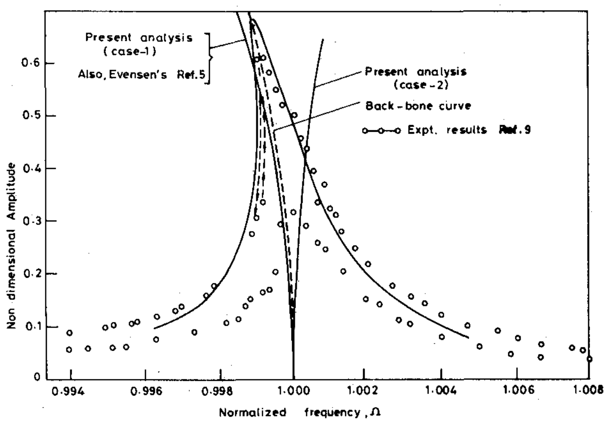

Fig. 2 Comparison with Chen's shell ($\xi = 0.2597$, $\mathcal{E} = 0.0076685$, $\nu = 0.31$).

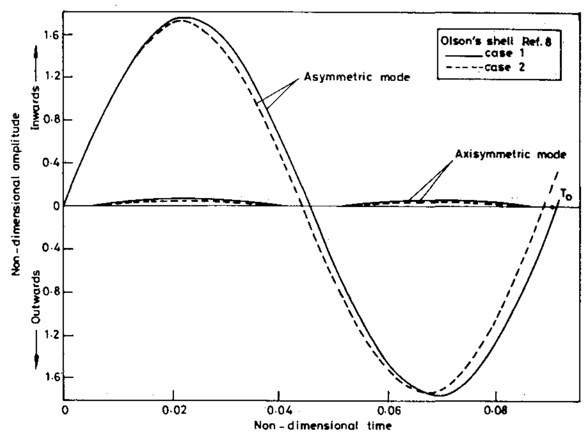

Fig. 3 Asymmetric and axisymmetric Modes—Olson's Shell.

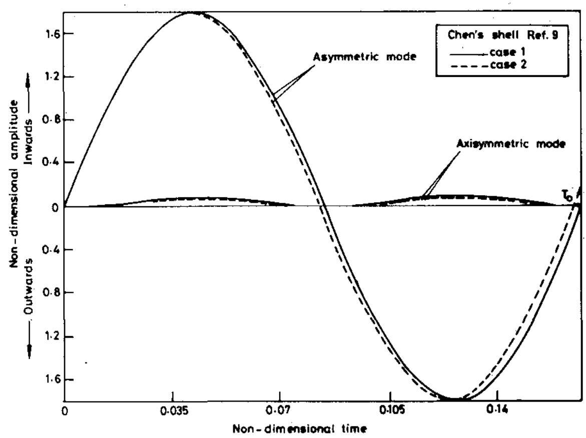

Fig. 4 Asymmetric and axisymmetric Modes—Chen's Shell.

The present results are compared with the available experimental results[8,9] in Figs. 1 and 2. Also, the numerical results are calculated using Evensen's[5] solution and plotted in these figures. Case 1 and case 2 correspond to the analysis based on the assumed mode shapes given by Eqs. (3) and (4), respectively. The independent $\sin^2(m\pi x/L)$ term for the axisymmetric part of the mode shape is found to give qualitatively better results than the other mode shape $\sin(\pi x/L)$. Results based on Evensen's analysis[5] agree well with the experimental work. It may be recalled that the mode shape chosen in Ref. 5 was such that the continuity condition in v was satisfied a priori. The present study also lends support to the contention[10] that the terms pertaining to the axisymmetric mode should be proportional to the square of those of the asymmetric mode. The axisymmetric and asymmetric parts of the assumed transverse mode shape obtained from the numerical integration of the modal equations are shown in Figs. 3 and 4. The double frequency contraction of the axisymmetric mode is seen for both cases under study, as was observed in Refs. 8 and 9. It is seen from these figures that even the small change in the amplitude of the axisymmetric mode due to the difference in the assumed axisymmetric part of the mode shapes affects the nonlinear coupling between the asymmetric and axisymmetric terms, resulting in a different qualitative nonlinear behavior.

The independent $\sin^2(m\pi x/L)$ term for the axisymmetric part of the mode shape is found to give better results than the other mode shape considered, $\sin(\pi x/L)$. The axisymmetric mode in the assumed deflection shape plays an important role in the nonlinear behavior of the circular cylindrical shell. With these observations, a good finite-element model that accurately encompasses the physics of the problem should be able to predict the nonlinear behavior of the shell for practical ranges of aspect ratio $\xi = (m\pi a/nL)$ and thickness parameter $\mathcal{E} = (n^2h/a^2)$.

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